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The Thouless–Anderson–Palmer equation for an analogue neural network with temporally fluctuating white synaptic noise

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Abstract

Effects of synaptic noise on the retrieval process of associative memory neural networks are studied from the viewpoint of neurobiological and biophysical understanding of information processing in the brain. We investigate the statistical mechanical properties of stochastic analogue neural networks with temporally fluctuating synaptic noise, which is assumed to be white noise. Such networks, in general, defy the use of the replica method, since they have no energy concept. The self-consistent signal-to-noise analysis (SCSNA), which is an alternative to the replica method for deriving a set of order parameter equations, requires no energy concept and thus becomes available in studying networks without energy functions. Applying the SCSNA to stochastic networks requires the knowledge of the Thouless–Anderson–Palmer (TAP) equation which defines the deterministic networks equivalent to the original stochastic ones. The study of the TAP equation which is of particular interest for the case without energy concept is very less, while it is closely related to the SCSNA in the case with energy concept. This paper aims to derive the TAP equation for networks with synaptic noise together with a set of order parameter equations by a hybrid use of the cavity method and the SCSNA.

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1. Introduction

The replica method [1] for random spin systems has been successfully employed in neural network models of associative memory [2–5]. However the replica method requires the concept of free energy. On the other hand, various types of neural network models which have no energy concept, such as a network with asymmetric synaptic coupling or temporally fluctuating synaptic coupling, may be in existence. The self-consistent signal-to-noise analysis (SCSNA) [6–9], which is an alternative approach to the replica method for deriving a set of order parameter equations, requires no energy concept. Thus it can be applicable to study the statistical properties of a wider class of networks including networks without energy concept.

The SCSNA, which was originally proposed for deriving a set of order parameter equations for a deterministic analogue neural network with or without energy functions, becomes applicable to a stochastic network by noting that the TAP equation defines the deterministic one equivalent to the original stochastic one. The SCSNA is closely related to the Thouless–Anderson–Palmer (TAP) equation [10, 11] via the concept of cavity method in the case where a network has energy concept [9] and the relationship between the two was studied in detail in the networks with two-body and multi-body interactions [9, 12]. The coefficient of the Onsager reaction term characteristic to the TAP equation, which determines the form of the transfer function, is self-consistently obtained through the concept shared by the cavity method and the SCSNA.

The TAP equation is of our interest for studying the statistical properties of a network without energy concept. However the TAP equations for a network with synaptic noise are not found in the literature. The main target of this paper is to derive the TAP equation for a network with temporally fluctuating synaptic noise as *multiplicative noise*.

The effects of such synaptic noise on the retrieval properties of networks have been studied in some recent works [13–16]. According to the stochastic resonance theory [17–19], the temporally fluctuating synaptic noise may possibly be expected to reduce the interference of the uncondensed patterns on the retrieval property of the network (noise in terms of stochastic resonance) and, as a result, enhance the retrieval property or the storage capacity (signal) of the network. However such an argument is not found in the literature. To discuss the effect of temporally fluctuating synaptic noise on the retrieval property, it is important to construct a tractable model to study the role played by such synaptic noise.

In literature, the term 'synaptic noise' is used in three kinds of meanings: (i) quenched disorder in synaptic couplings [20], (ii) randomness related to the dilution of synaptic couplings [21] and (iii) temporal fluctuation of synaptic couplings [13–16]. We will use the term 'synaptic noise' in the third meaning in the present paper.

Cortes *et al* [25] (see also [16]) investigated the case where neurons obey a master equation with continuous time and the synaptic couplings obey (i) slow dynamics [22, 23], (ii) fast dynamics [15, 16, 24, 25] and (iii) middle speed dynamics [26, 27] compared to the dynamics of neurons. In the first case, since the synaptic couplings obey slow dynamics, the adiabatic approximation for the synaptic couplings becomes exact in the limit where the time scale of synapse dynamics $\tau \to \infty$ and quenched random noise in couplings again arises. Thus the synaptic noise is regarded as the well-known quenched random variable and this type of synaptic noise has been studied as 'synaptic noise' in many publications [28].

In the second case, since the dynamics of the synaptic noise is sufficiently fast compared to the dynamics of neurons, one can define the effective strength of synaptic coupling by averaging the temporally fluctuating synaptic coupling [13–16, 29]. In this case, one can find the effective Hamiltonian of the network and use well-known replica method [1] to obtain the order parameters analytically [2, 3]. Another example of the fast synaptic dynamics can be found in [30], which studies the properties of the equilibrium state of the system with stochastically evolving couplings.

On the other hand, the third case is difficult to deal analytically especially in the case where the number of memory patterns is proportional to the total number of neurons and only numerical results based on computer simulations exist [26, 27]. In spite of these recent efforts to elucidate the effects of synaptic noise on the retrieval properties of neural networks, such preceding studies have been based on the macroscopic viewpoint, where the order parameters solely have been investigated, and the TAP equations for such cases have not been reported.

The purpose of this paper is two-fold. (i) We will derive the TAP equation for a stochastic analogue network with temporally fluctuating multiplicative synaptic noise which is not found in the literature. (ii) We will study the SCSNA and the TAP equation for such a network to elucidate the effects of the multiplicative synaptic noise on the retrieval property from both microscopic and macroscopic viewpoints. A part of this work is reported elsewhere [31].

This paper is organized as follows: in the next section, we will describe an analogue neural network model with temporally fluctuating synaptic noise as multiplicative noise which is assumed to be white noise to write down a set of Langevin equations, and derive the corresponding Fokker–Planck equation. We will see that the equilibrium solution of the Fokker–Planck equation is given as a Gibbs probability density with the effective temperature, which should be determined self-consistently in the thermodynamic limit. In section 3, we will apply the cavity method to derive the formal expression of the TAP equation (pre-TAP equation). Then using the SCSNA, we will self-consistently obtain the concrete form of the transfer function which yields the complete form of the TAP equation as well as a set of order parameter equations. In section 4, the phase diagram for our model will be shown. In the last section, we will conclude this paper.

2. Model and Fokker–Planck equation formalism

Let us deal with the following stochastic analogue neural network of N neurons with temporally fluctuating synaptic noise:

$$\dot{x}_i = -\phi'(x_i) + \sum_{j(\neq i)} J_{ij}(t)x_j + \eta_i(t), \tag{1a}$$

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t'), \tag{1b}$$

where x_i $(i = 1, \ldots, N)$ represents a state of the neuron at site i taking a continuous value, $\phi(x_i)$ is a potential of an arbitrary form which determines the probability distribution of x_i in the case without the input $\sum_{j(\neq i)} J_{ij}x_j$, η_i is the Langevin white noise with its noise intensity $2D$ and $J_{ij}(t)$ is the synaptic coupling. We note here that, in the case of associative memory neural network, the synaptic coupling J_{ij} is usually defined by the well-known Hebb learning rule. However some experimental results show that the synaptic couplings have temporal fluctuations which originate from the dynamics of neurotransmitters or kinetics of ion channels independent of that of neurons [32], and hence the effects of such synaptic noise may be relevant to the retrieval properties in realistic networks. To investigate such effects of synaptic noise, we assume the synaptic coupling taking the form

$$J_{ij}(t) = \bar{J}_{ij} + \epsilon_{ij}(t), \tag{2a}$$

$$\langle \epsilon_{ij}(t)\epsilon_{kl}(t')\rangle = \frac{2\tilde{D}}{N}\delta_{ik}\delta_{jl}\delta(t-t'), \tag{2b}$$

where $\bar{J}_{ij}$ is defined by the usual Hebb learning rule $\bar{J}_{ij} \equiv \frac{1}{N}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu$ with $p = \alpha N$ the number of patterns embedded in the network, $\xi_i^\mu = \pm 1$ is the μth embedded pattern at neuron i and $\epsilon_{ij}(t)$ denotes the synaptic noise independent of $\eta_i(t)$, which we assume in our model as white noise with its intensity $2\tilde{D}/N$ for simplicity. Note that, in equation (1b), the synaptic noise behaves as *multiplicative noise* and the synaptic coupling $J_{ij}(t)$ is asymmetric.

Noting

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}s\int_t^{t+\Delta t}\mathrm{d}s'\left\langle\sum_{k(\neq i)}\epsilon_{ik}(s)x_k(s)\sum_{l(\neq j)}\epsilon_{jl}(s')x_l(s')\right\rangle=\frac{2\tilde{D}}{N}\delta_{ij}\sum_{k(\neq i)}x_k^2$$

by means of Ito integral, we obtain the Fokker–Planck equation corresponding to the Langevin equation (1*a*) as

$$\frac{\partial P(t,\mathbf{x})}{\partial t}=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left\{-\phi'(x_i)+\sum_{j(\neq i)}\bar{J}_{ij}x_j-(D+\tilde{D}\hat{q})\frac{\partial}{\partial x_i}\right\}P(t,\mathbf{x}),\qquad(3)$$

where $\hat{q}\equiv\frac{1}{N}\sum_{j(\neq i)}x_j^2$. Since the self-averaging property holds in the thermodynamic limit $N\to\infty$, one can identify $\hat{q}$ as

$$\hat{q}=\frac{1}{N}\sum_{i=1}^{N}\langle x_i^2\rangle,\qquad(4)$$

where $\langle\cdot\rangle$ represents the thermal average with respect to $P(t,\mathbf{x})$. Thus equation (3) is found to be a nonlinear Fokker–Planck equation whose diffusion coefficient $D+\tilde{D}\hat{q}$ depends on the probability density $P(t,\mathbf{x})$ [33, 34]. In this paper, we are concerned with deriving the TAP equation and order parameter equations for the equilibrium state self-consistently. Furthermore the order parameter $\hat{q}$ is also obtained self-consistently in our framework as shown below. Supposing $\hat{q}$ is given, the Fokker–Planck equation (3) turns to be a linear equation and one can easily find the equilibrium probability density for the linear Fokker–Planck equation (3) as

$$P_N(\mathbf{x})=Z^{-1}\exp\left\{-\beta_{\mathrm{eff}}\left(\sum_{i=1}^{N}\phi(x_i)-\sum_{i<j}\bar{J}_{ij}x_ix_j\right)\right\},\qquad(5)$$

where Z denotes the normalization constant and

$$\beta_{\mathrm{eff}}^{-1}\equiv D+\tilde{D}\hat{q}\qquad(6)$$

plays the role of the *effective temperature of the network*. Note that the temperature of the system is modified to $\beta_{\mathrm{eff}}^{-1}$ as a consequence of the multiplicative noise and it depends on the order parameter $\hat{q}$. Here it is easily checked that the equilibrium distribution of the system becomes Gibbs distribution in the thermodynamic limit $N\to\infty$.

Since we have explicitly written down the equilibrium probability density (5) as a Gibbsian form, one can define the (effective) Hamiltonian of N-body system as

$$H_N\equiv\sum_{i=1}^{N}\phi(x_i)-\sum_{i<j}\bar{J}_{ij}x_ix_j.\qquad(7)$$

Then regarding the original network with multiplicative noise (1*a*) as an analogue version of the standard Hopfield model whose Hamiltonian is given by equation (7) with the effective temperature $\beta_{\mathrm{eff}}^{-1}$, one can apply the usual cavity method [2] to this system and derive the (pre-)TAP equation.

3. The cavity method and self-consistent signal-to-noise analysis

We have obtained the equilibrium probability density as a Gibbsian form (5) and the effective Hamiltonian (7) in the previous section. Thus the cavity method [2], which is usually applied to the network models for deriving the TAP equation [8, 9], is applicable for our model. According to the cavity method, we divide the Hamiltonian of N-body system (7) into that of $(N-1)$-body system and the part involving the state of ith neuron as

$$H_N = \phi(x_i) - h_i x_i + H_{N-1},$$

where $h_i \equiv \sum_{j(\neq i)} \bar{J}_{ij} x_j$ is the local field at site i and the Hamiltonian of $(N-1)$-body system H_{N-1} is given as $H_{N-1} \equiv \sum_{j(\neq i)} \phi(x_j) - \sum_{j<k(\neq i)} \bar{J}_{jk} x_j x_k$. Then the marginal probability density distribution of x_i and the local field h_i is given as

$$\begin{aligned} P_N(x_i, h_i) &= \int \left[\prod_{j(\neq i)} \mathrm{d}x_j\right] \delta\left(h_i - \sum_{j(\neq i)} \bar{J}_{ij} x_j\right) P_N(\mathbf{x}) \\ &= \tilde{Z}^{-1} \exp\{-\beta_{\mathrm{eff}}[\phi(x_i) - h_i x_i]\} P_{N-1}(h_i), \end{aligned}$$

where $\tilde{Z}$ is the normalization constant and $P_{N-1}(h_i)$ denotes the probability density of the local field h_i in the $(N-1)$-body system defined as

$$P_{N-1}(h_i) \equiv Z_{N-1}^{-1} \int \left[\prod_{j(\neq i)} \mathrm{d}x_j\right] \delta\left(h_i - \sum_{j(\neq i)} \bar{J}_{ij} x_j\right) \exp[-\beta_{\mathrm{eff}} H_{N-1}],$$

where Z_{N-1} denotes the normalization constant. Since the local field is given as the summation of a sufficiently large number of random variables and their cross-correlations are expected to be $\mathrm{O}(1/\sqrt{N})$, one can expect that $P_{N-1}(h_i)$ turns out to be a Gaussian density in the thermodynamic limit $N \to \infty$ according to the central limit theorem:

$$P_{N-1}(h_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(h_i - \langle h_i \rangle_{N-1})^2}{2\sigma^2}\right],$$

where $\langle\cdot\rangle_{N-1}$ represents the thermal average with respect to the $(N-1)$-body probability density $P_{N-1}(\mathbf{x})$ and σ^2 is the variance of $P_{N-1}(h_i)$, which is evaluated later self-consistently in the framework of the SCSNA. Then taking the average of x_i with respect to the marginal probability $P_N(x_i, h_i)$ straightforwardly yields

$$\langle x_i \rangle = F(\langle h_i \rangle_{N-1}), \tag{8}$$

where F is a transfer function defined as

$$F(y) \equiv \frac{\int \mathrm{d}x\, x \exp\left\{-\beta_{\mathrm{eff}}\left[\phi(x) - yx - \frac{\beta_{\mathrm{eff}}\sigma^2}{2} x^2\right]\right\}}{\int \mathrm{d}x \exp\left\{-\beta_{\mathrm{eff}}\left[\phi(x) - yx - \frac{\beta_{\mathrm{eff}}\sigma^2}{2} x^2\right]\right\}}. \tag{9}$$

Similarly $\langle h_i \rangle_{N-1}$ is obtained as

$$\langle h_i \rangle_{N-1} = \langle h_i \rangle - \beta_{\mathrm{eff}} \sigma^2 \langle x_i \rangle.$$

Thus we have the pre-TAP equation

$$\langle x_i \rangle = F\left(\sum_{j(\neq i)} \bar{J}_{ij} \langle x_j \rangle - \Gamma_{\mathrm{Ons}} \langle x_i \rangle\right), \tag{10}$$

where $\Gamma_{\mathrm{Ons}} \equiv \beta_{\mathrm{eff}}\sigma^2$. Since the concrete form of the transfer function F depends on the effective temperature β_{eff} and the variance of the local field σ^2, it is necessary to obtain β_{eff} and σ^2 to have the TAP equation [7, 9].

Equation (10) is regarded as defining a deterministic analogue network corresponding to the original stochastic one (1a), and hence we can apply the SCSNA to equation (10) to determine β_{eff} and σ^2 self-consistently as was studied for the case without synaptic noise

[7, 9]. For simplicity, we here assume that the only one condensed pattern $\{\xi_i^1\}$ is retrieved. The extension to the case of an arbitrary finite number of condensed patterns is straightforward. Using the overlap order parameter $m^\mu \equiv \frac{1}{N}\sum_{i=1}^N \xi_i^\mu \langle x_i\rangle$, the equilibrium average of the local field is rewritten as

$$\langle h_i\rangle = \xi_i^1 m^1 + \sum_{\mu\geqslant 2} \xi_i^\mu m^\mu - \alpha\langle x_i\rangle. \tag{11}$$

Using the SCSNA, the above local field can be rewritten as [8, 9]

$$\langle h_i\rangle = \xi_i^1 m^1 + \xi_i^\mu m^\mu + z_{i\mu} + \Gamma_{\mathrm{SCSNA}}\langle x_i\rangle, \tag{12}$$

where $\sum_{\nu\geqslant 2}\xi_i^\nu m^\nu = \xi_i^\mu m^\mu + z_{i\mu} + \gamma\langle x_i\rangle$, $\Gamma_{\mathrm{SCSNA}} \equiv \gamma - \alpha$ and $z_{i\mu}$ is a Gaussian random variable with a zero mean. As shown below, we will evaluate the overlap m^μ self-consistently, and then obtain $z_{i\mu}$ and γ through the equivalence between the expression of the local field (11) and (12). Substituting equation (12) into the pre-TAP equation (10) reads

$$\langle x_i\rangle = F\left(\xi_i^1 m^1 + \xi_i^\mu m^\mu + z_{i\mu} + (\Gamma_{\mathrm{SCSNA}} - \Gamma_{\mathrm{Ons}})\langle x_i\rangle\right)$$

and comparing this equation with equation (8) yields [9]

$$\Gamma_{\mathrm{SCSNA}} = \Gamma_{\mathrm{Ons}},$$

since $\langle h_i\rangle_{N-1}$ is considered to be a Gaussian random variable which should not contain the Onsager reaction term. Noting that $m^\mu = \mathrm{O}(1/\sqrt{N})$ for $\mu \geqslant 2$, one can obtain the overlap for uncondensed patterns as

$$m^\mu = \frac{1}{N(1-U)}\sum_{j=1}^N \xi_j^\mu F\left(\xi_j^1 m^1 + z_{j\mu}\right), \tag{13a}$$

$$U \equiv \frac{1}{N}\sum_{j=1}^N F'\left(\xi_j^1 m^1 + z_{j\mu}\right), \tag{13b}$$

where F' denotes the derivative of the transfer function F, and the order expansion of F with respect to $1/\sqrt{N}$ has been applied to $\langle x_i\rangle = F\left(\xi_i^1 m^1 + z_{i\mu} + \xi_i^\mu m^\mu\right)$. Using equation (13a) and the definitions of $z_{i\mu}$ and γ, one finds

$$\gamma = \frac{\alpha}{1-U},$$

$$z_{i\mu} = \frac{1}{N(1-U)}\sum_{\nu(\neq 1,\mu)}\sum_{j(\neq i)} \xi_i^\nu \xi_j^\nu F\left(\xi_j^1 m^1 + z_{j\nu}\right).$$

Thus the variance of $z_{i\mu}$ is evaluated as

$$\sigma_z^2 = \frac{\alpha}{(1-U)^2}\langle F^2(\xi m^1 + z)\rangle_{\xi,z}, \tag{14a}$$

where $\langle\cdot\rangle_{\xi,z}$ represents the average over random variables $\xi = \pm 1$ and the Gaussian variable z, and the self-averaging property has been used. Similarly one obtains the set of order parameter equations as

$$m^1 = \langle \xi F(\xi m^1 + z)\rangle_{\xi,z}, \tag{14b}$$

$$U = \langle F'(\xi m^1 + z)\rangle_{\xi,z}, \tag{14c}$$

$$\Gamma_{\mathrm{Ons}} = \Gamma_{\mathrm{SCSNA}} = \frac{\alpha U}{1-U}. \tag{14d}$$

In the case where the multiplicative synaptic noise does not exist or the intensity of the synaptic noise is zero, i.e., $\beta_{\rm eff} = \beta \equiv 1/D$, the set of order parameter equations (14*a*)–(14*d*) takes a closed form and determines the form of the transfer function F as well as the order parameters self-consistently. For the case with multiplicative noise, however, it does not suffice to determine the form of the transfer function. We need the order parameter $\hat{q}$, which determines $\beta_{\rm eff}$, as well as $m^1, U, \sigma_z^2, \Gamma_{\rm Ons}$ to determine the concrete form of F. Since $\hat{q}$ is related to the macroscopic susceptibility of the system and, by definition of F (9), the order parameter U corresponds with the susceptibility as $U = \beta_{\rm eff}(\langle x^2\rangle - \langle x\rangle^2)$, one finds

$$\hat{q} = \frac{U}{\beta_{\rm eff}} + \frac{(1-U)^2}{\alpha}\sigma_z^2. \tag{14e}$$

The set of equations (6), (14*a*)–(14*e*) takes a closed form and thus one can determine the form of F self-consistently as well as the set of order parameters. Therefore substituting into the pre-TAP equation (10), the solutions $\beta_{\rm eff}$ and $\Gamma_{\rm Ons}$ that are self-consistently obtained within this framework yield the TAP equation.

4. Phase diagram and numerical results

We have derived the TAP equation as well as the set of order parameter equations in the previous section. In this section, we show the phase diagram by solving the set of order parameter equations (14*a*)–(14*e*) numerically and investigate the effect of the multiplicative synaptic noise.

For the well-known transfer function of the Ising neurons $F(x) = \tanh(\beta x)$, it is easy to understand the effects of the interference of the synaptic noise. This choice of the transfer function is equivalent to taking the potential ϕ as

$$\frac{\exp[-\beta_{\rm eff}\phi(x)]}{\int \exp[-\beta_{\rm eff}\phi(x)]\,{\rm d}x} = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1).$$

In the Ising neuron model, since $\hat{q}$ is simply given as $\hat{q} = 1$ and $\beta_{\rm eff}^{-1} = D + \tilde{D}$, the retrieval state vanishes for $\tilde{D} \geqslant 1$ according to the results of Amit–Geutfreund–Sompolinsky (AGS) [3].

In this section, for simplicity, we consider the double-well potential whose minima are located at $x = \pm 1$:

$$\phi(x) = \frac{A}{4}x^4 - \frac{A}{2}x^2, \tag{15}$$

where A determines the depth of the wells of the potential. This potential yields a continuous distribution of neuron states and thus defines an *analogue* network model in which $\hat{q}$ is non-trivial. We investigate a phase diagram for the *analogue* network model and elucidate the effects of the multiplicative noise on the retrieval properties.

Figure 1 illustrates the storage capacity α as a function of the intensity of the external noise D. The paramagnetic-spin glass phase boundary for $\tilde{D} = 0$ is also displayed (solid thick line). The solid line is for the absence of the synaptic noise, i.e., $\tilde{D} = 0$. The line in figure 1 denotes the numerical solution of the order parameter equations for each intensity of the synaptic noise $\tilde{D} = 0, 0.2, 0.4, 0.6$ and the retrieval state vanishes at $\tilde{D} \sim 1.05$ for $A = 20$. The α-D line for the *analogue network* is deformed compared to the Ising networks. This is the effect of potential properties and the *effective temperature*, or the *non-trivial order parameter* $\hat{q}$, while $\hat{q} = 1$ for Ising neurons.

We can see that the storage capacity is incrementally decreased as the intensity of the synaptic noise increases. This result is reasonable since the memories are encoded in the

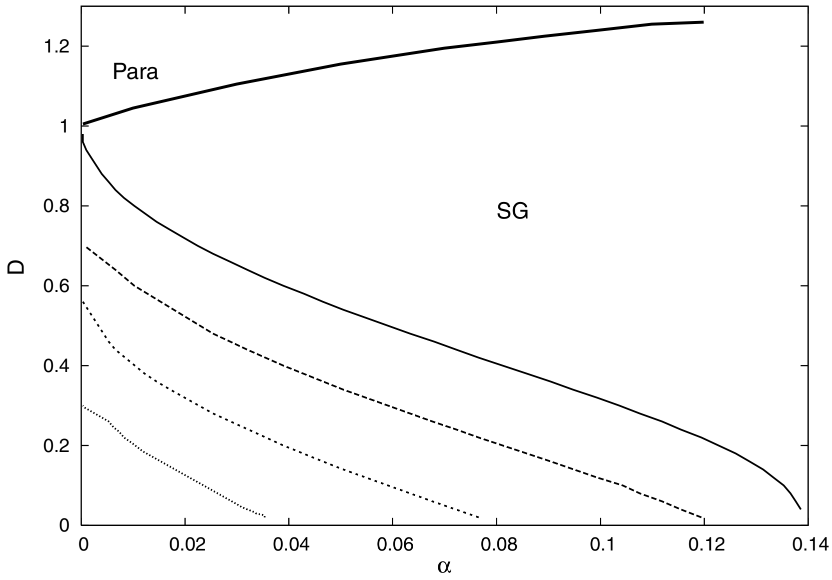

Figure 1. Storage capacity α as a function of the intensity of the external noise D for various values of $\tilde{D}$. The paramagnetic-spin glass (SG) phase boundary for $\tilde{D} = 0$ is also displayed (solid thick line). The solid curve denotes the storage capacity for $\tilde{D} = 0$. The broken (– – –), dashed (- - - -) and dotted (· · · · · ·) curves represent the storage capacity for $\tilde{D} = 0.2, 0.4, 0.6$, respectively. The retrieval state locates below the curve and vanishes at $\tilde{D} \sim 1.05$. We set $A = 20$.

synaptic coupling as local minima of the *effective* free energy corresponding to equation (5) and the synaptic noise is expected to disturb the fine structure of the energy landscape.

Figure 2 displays the α-dependence of the overlap m^1 obtained from the SCSNA together with that from numerical simulations with $N = 3000$. We can see that the overlap m^1 decreases as the number of embedded patterns α increases and retrieval state vanishes ($m^1 = 0$) at $\alpha_c \sim 0.049$.

Figure 3 illustrates the distribution of the thermal average of the state of neurons, or the local magnetization $\langle x_i \rangle$ at $D = 0, \tilde{D} = 0.5, \alpha = 0.1, A = 20$ obtained from numerical simulations with $N = 3000$. We can see from figure 2 that the overlap $m^1 = 0$ in this regime. However the local magnetizations $\langle x_i \rangle$ s are seen to distribute around $\langle x_i \rangle = \pm 1$. This means that a spin glass phase arises in this regime. We can also show the existence of the non-retrieval spin glass phase analytically by solving the set of order parameter equations (6), (14*a*)–(14*e*). Since $m^1 = 0$ in the non-retrieval phase, the order parameter equations (14*a*) and (14*c*) become

$$\sigma_z^2 = \frac{\alpha}{(1-U)^2} \langle F^2(z) \rangle_z, \tag{16a}$$

$$U = \langle F'(z) \rangle_z, \tag{16b}$$

where $\langle \cdot \rangle_z$ denotes the average with respect to the Gaussian random variable z. For the non-retrieval phase $m^1 = 0$, it is trivial that the order parameter equation $m^1 = 0 = \langle \xi F(z) \rangle_{\xi,z}$ holds. By definition of the transfer function F, the order parameter U is rewritten as

$$U = \beta_{\text{eff}}(\hat{q} - q), \tag{16c}$$

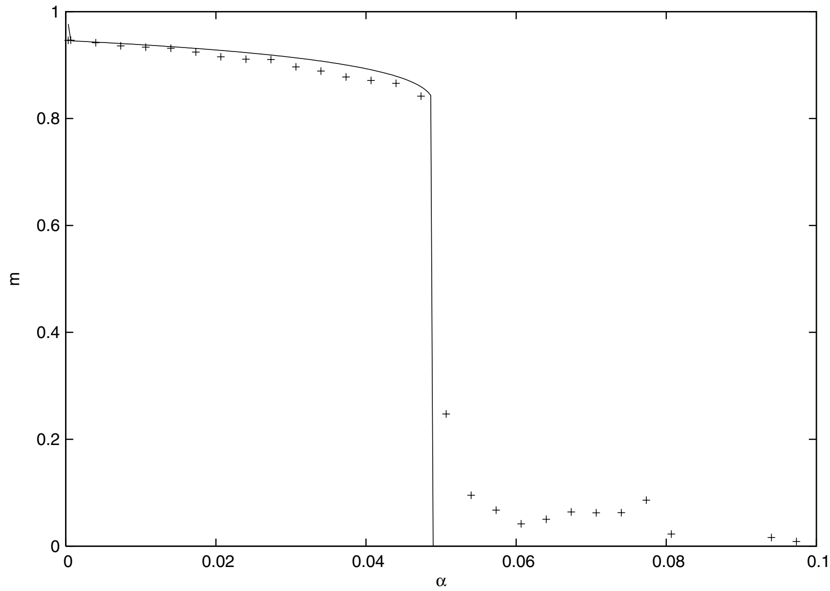

Figure 2. α-dependence of the overlap m^1 obtained from the SCSNA (solid curve) together with that from numerical simulations with $N = 3000$ (dots). The potential is given by equation (15) with $A = 20$. The intensities of additive and multiplicative synaptic noise are $D = 0$ and $\tilde{D} = 0.5$ respectively.

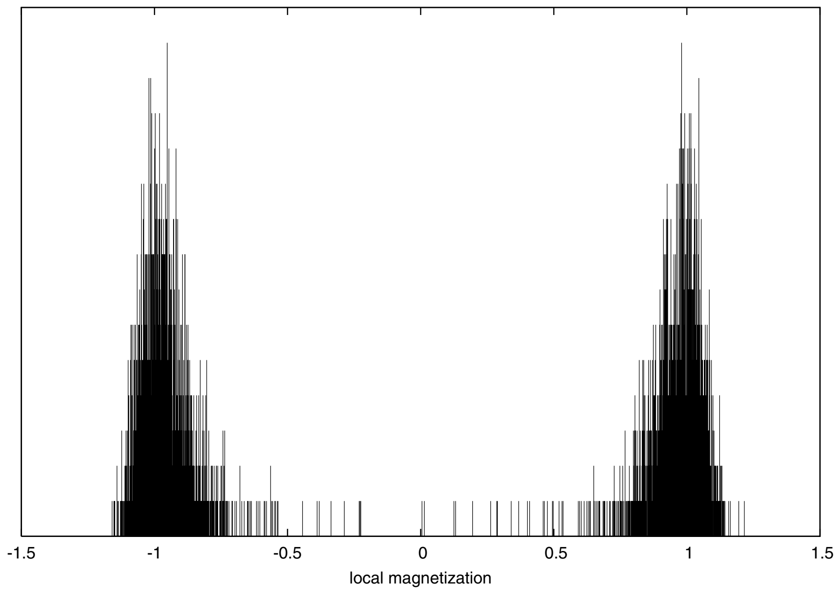

Figure 3. Distribution of the thermal average of the state of neurons, or the local magnetization with parameters $D = 0$, $\tilde{D} = 0.5$, $\alpha = 0.1$, $A = 20$. From figure 2 this parameter set locates in the non-retrieval phase. This distribution shows that the non-retrieval spin glass phase arises in this regime.

where $q = \frac{1}{N}\sum_i \langle x_i \rangle^2$ is the Edward–Anderson order parameter. The set of order parameter equations (6), (14*d*), (14*e*), (16*a*)–(16*c*) takes a closed form. Thus we can find the non-retrieval spin glass phase by solving these equations to obtain $q \neq 0$. Since the Edward–Anderson order parameter q is expected to be small in the regime close to the paramagnetic phase, the Taylor expansion with respect to q is applicable for these order parameter equations to illustrate the paramagnetic ($q = 0$)-spin glass ($q \neq 0$) phase boundary (see figure 1). This boundary is expected to correspond to the de Almeida-Thouless (AT) line. The study on the relationship between the SCSNA and the replica symmetry breaking is underway.

5. Concluding remarks

We have derived the TAP equation for a stochastic analogue neural network with temporally fluctuating multiplicative synaptic noise, which is not found in the literature. More specifically, we have derived the TAP equation together with the set of order parameter equations by using the SCSNA and the cavity method. Our original model *does not* have the concept of free energy. Since the self-averaging property holds in the thermodynamic limit $N \to \infty$, we have found that the nonlinear Fokker–Planck equation (3) becomes quasi-linear to allow one to obtain equilibrium probability density obeying the Gibbs one with effective temperature and hence that the network with white synaptic noise has the *effective* Hamiltonian in the large-N limit. Thus the cavity method, which is applicable to the model with energy concept, becomes available to obtain the (pre-)TAP equation. Unlike the case without synaptic noise, the concrete form of the transfer function F of our model has been found to depend not only on the coefficient of the Onsager reaction term but also on the order parameter $\hat{q}$. $\hat{q}$ as well as the coefficient of the Onsager reaction term has been obtained self-consistently within the framework of the SCSNA. The full TAP equation straightforwardly follows from the pre-TAP equation by substituting the solutions of the order parameter equations into the pre-TAP equation (10).

Furthermore, we have found that the storage capacity of the network gradually decreases as the intensity of the synaptic noise increases, since the fine structure of the energy landscape tends to disappear by the interference of the synaptic noise. This effect of the interference of the synaptic noise on the behavior of the retrieval property has been shown to appear via the effective temperature $\beta_{\rm eff}^{-1} \geqslant D$.

All the results presented in this paper are obtained via the cavity method and the SCSNA. On the other hand, the order parameter equations (14*a*)–(14*e*) can be reproduced as the replica symmetric case by the replica method, since the system has the effective Hamiltonian (7). Thus our results are expected to be exact within the replica symmetric approximation. However, the development of the analysis in the framework of the SCSNA for replica symmetry breaking solutions is now underway.

In other works dealing with the temporal fluctuation in synaptic couplings [14, 15], the authors study the case of the fast synapse dynamics. Then the synaptic coupling is modified to take the form of ‘effective synaptic coupling’ and the system becomes to have an effective Hamiltonian. In this case the ‘effective synaptic coupling’ is *straightforwardly determined* by both the number of embedded patterns and the intensity of Langevin noise associated with neuron dynamics. On the other hand, the time scale of fluctuation of synaptic coupling in our model is *comparable to that of the neuron dynamics*. Our model results in having the ‘effective temperature’ and hence the effective Hamiltonian in the thermodynamic limit. However, in our model, the ‘effective temperature’ is *determined only self-consistently* together with the other order parameters.

In this paper, we have dealt with a network subjected to asymmetric multiplicative synaptic noise given as white noise involving both pre- and post-neuron, and the noise has no correlation with the synaptic coupling given by the Hebb learning rule. However some other versions of synaptic noise may be considered: (i) synaptic noise depending only on pre- or post-neuron, (ii) synaptic noise correlated with the Hebb learning rule and (iii) coloured synaptic noise. For some of these cases, one can rigorously derive the TAP equation and the set of order parameter equations similarly to the case we have seen in this paper. The analysis for such cases will be reported elsewhere.

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